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Directed true self-avoiding Lévy flights

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Abstract. The critical exponent and the exponent of its logarithmic corrections for the end-to-end displacement of directed Lévy flights are calculated. The probability distribution with the decay $x^{-\mu}(\ln x)^{-\kappa}$ for the step-length $|r|>x$, which is more general than the usual one $x^{-\mu}$, is considered. The self-avoidance and hence the dimension d have no influence on the exponents. This is shown rigorously for $d\leqslant 2$ and seems to be true also for $d>2$. Three different definitions for the critical behaviour are given, one of them is especially appropriate for computer simulations.

1. Introduction

A random flight is similar to a random walk, except that it has a probability distribution of step-length instead of a fixed step size. Mandelbrot discussed certain random flights with a probability distribution of the type

$$\Pr\{r\colon r>x\}\propto x^{-\mu} \qquad 0<\mu<2 \tag{1}$$

for large x and called them Lévy flights [1] since this probability distribution is one of Lévy type. Generally, we define a Lévy flight as a flight with the probability distribution that belongs to the domain of attraction of a stable (Lévy) distribution [2].

The trajectories of Lévy flights differ strikingly from that of ordinary random walks and flights. The trace of the sites visited by an ordinary Lévy-type propagator forms a fractal set with Hausdorff dimension [3] μ instead of 2. Because of the divergence of the variance of step size, the asymptotic behaviours of various Lévy flights are all different from that of random walks. For instance, the critical correlation exponent ν of a self-avoiding Lévy flight (SALF) [4], defined via the geometric average of the end-to-end displacement

$$\langle \ln R(N)^2\rangle \sim 2\nu \ln N \tag{2}$$

for large number N of steps, is no longer that of a self-avoiding walk (SAW), i.e. $\nu=3/(2+d)$ for $d\leqslant d_c=4$ [5]. Following the analysis of the equivalent n-vector spin model presented by de Gennes [5], Halley and Nakanshi [6] obtained the ε-expansion of the exponent υ of the node-avoiding Lévy flight (NALF) as

$$\nu=\frac{1}{\mu}\left(1+\frac{1}{4\mu}\varepsilon+(19-\tfrac{5}{4}\mu^2)\varepsilon^2/(64\mu^2)+\ldots\right) \tag{3}$$

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with $\varepsilon = d - 2\mu$, which was considered to coincide with simulation results more precisely than the exponents determined with a Flory-type argument [4]

$$\nu = \begin{cases} 3/(\mu+d) & \text{for } d \leqslant d_c = 2\mu \\ 1/\mu & \text{for } d > d_c. \end{cases} \tag{4}$$

Even the two types of SALFs, the NALF and the path-avoiding Lévy flight (PALF), are in different universality classes [7, 8] under a certain upper marginal dimension d_c.

Recently it has been realized that the introduction of a global bias in geometrical models like directed percolation and directed SAWs [9–11] leads to novel anisotropic critical behaviours. How about the critical behaviour of directed self-avoiding flights [12]? Does the directed self-avoiding Lévy flight (DSALF) belong to the same universality class as either the directed SAW or the ordinary SALFs? From the results in [7] one can get the critical exponent ν of the PALF in 1D. But how do this exponent and the other related exponents $\nu_\parallel$ and $\nu_\perp$ depend on the spatial dimension? In this paper we first discuss the influence of self-avoidance on the directed flights and then, with the help of some useful definitions of the exponents, we calculate the critical exponents of the DSALF with the probability for steps $|r| > x$ with decay $x^{-\mu}(\ln x)^{-\kappa}$, $0 \leqslant \mu \leqslant 2$. With the help of an example, we also show that the critical exponents can be determined even if the probability distribution does not lie in the domain of attraction of a Lévy distribution.

2. Self-avoidance of directed flights

Let us consider a directed flight in the d-dimensional Euclidean space $\mathbb{R}^d_+ \equiv \{(x_1, x_2, \ldots, x_d) | x_1 \geqslant 0\}$, where the probability distribution for the Nth step $r(N) \equiv (r_1(N), r_2(N), \ldots, r_d(N))$ has the probability density $p(r(N))$ with

$$\int_{r_1<0} p(r)\,\mathrm{d}^d r = 0 \qquad \text{and} \qquad \int_{r_1>0} p(r)\,\mathrm{d}^d r = P > 0 \qquad \text{if } r_1(N-1) > 0.$$

But if

$$r_1(N-k) = r_1(N-k+1) = r_1(N-1) = 0 \qquad \text{and} \qquad r_1(N-k-1) \neq 0$$

there can be with respect to $p(r(N))$ a positive probability, denoted as $q(r(N-k), \ldots, r(N-1)) \leqslant 1 - P$, with which the propagator visits a site more than once or the path between $R(N) \equiv \Sigma_{n=1}^{N} r(n)$ and $R(N-1)$ intersects that between $R(N-k)$ and $R(N-1)$. These steps with probability $q(r(N-k), \ldots, r(N-1))$ have to be cancelled for a directed node- or path-avoiding flight. Then the probability density $p(r(N))$ must be modified with some weight $Q(r(N); r(N-k), \ldots, r(N-1))$, which may be written for the 'true' self-avoiding flight [13–14] we will treat below, as

$$\{1 - q(r(N-k), \ldots, r(N-1))\}^{-1}$$

if $r(N)$ is allowed and 0 elsewhere. For the special case that $\int_{r_i<0} p(r)\,\mathrm{d}^d r$ is 0 for every i $(1 \leqslant i \leqslant d)$, we get the fully directed flight [10], which is always self-avoiding.

In statistical physics one is mainly interested in the asymptotical behaviours of $R(N)$ and of two related quantities—the two projections of $R(N)$, $R_\parallel(N)$ and $R_\perp(N)$, onto the directions parallel and perpendicular to the preferred direction, respectively. If the probability distribution satisfies

$$p(r_1, r_2, r_3, \ldots, r_d) = p(r_1, -r_2, -r_3, \ldots, -r_d) \tag{5a}$$

which describes symmetrical directed flights, the expectation value of the ith axial component of $R(N)$, $\langle R_i(N)\rangle$, is 0 for $2 \leq i \leq d$, i.e. the unit vector $e=(1,0,\ldots,0)$ gives the preferred direction. For fully directed flights with the other kind of symmetric probability distribution

$$p(r_1,\ldots,r_i,\ldots,r_j,\ldots,r_d)=p(r_1,\ldots,r_j,\ldots,r_i,\ldots,r_d) \tag{5b}$$

the preferred direction is obviously given by $e=(1,1,\ldots,1)/\sqrt{d}$. Generally, the preferred direction is defined by

$$e=\lim_{N\to\infty}\langle R(N)\rangle/|\langle R(N)\rangle|. \tag{6}$$

Then we have

$$R_{\|}(N)=(R(N)\cdot e)e \tag{7}$$

and

$$\boldsymbol{R}_{\perp}(N)=\boldsymbol{R}(N)-\boldsymbol{R}_{\|}(N) \tag{8}$$

where the dot denotes the scalar product. For directed ordinary flights (with finite variance) one can define the critical exponent ν_* for the asymptotical behaviour of $R_*(N)$ as

$$\nu_*=\lim_{N\to\infty}\ln\langle \boldsymbol{R}_*(N)^2\rangle/2\ln N \tag{9}$$

where $*$ represents either $\|$ or $\perp$ or no index. Pythagoras' law $\boldsymbol{R}(N)^2=\boldsymbol{R}_{\|}(N)^2+\boldsymbol{R}_{\perp}(N)^2$ implies

$$\nu=\max\{\nu_{\|},\nu_{\perp}\}. \tag{10}$$

For directed Lévy flights, the variance of $R_*(N)$ is diverging and we must define ν_* in another way. This will be discussed separately in section 3.

If

$$q(\boldsymbol{r}(N-k),\ldots,\boldsymbol{r}(N-1))=0 \qquad p(\boldsymbol{r}(N-k))\ldots p(\boldsymbol{r}(N-1)) \quad \text{a.s.} \tag{11}$$

(almost surely) then every step of the flight is independent, i.e. the flight is always self-avoiding. Equation (11) is fulfilled for the flights with

$$\int_{\boldsymbol{r}\cdot\boldsymbol{a}>0} p(\boldsymbol{r})\,\mathrm{d}^d\boldsymbol{r}=1 \tag{12}$$

for an arbitrary vector $\boldsymbol{a}$, e.g. the fully directed flights and the partially directed flights with $\int_{r_1>0} p(\boldsymbol{r})\,\mathrm{d}^d\boldsymbol{r}=P=1$. For the directed ordinary flights of this type, the critical exponents are

$$\nu=\nu_{\|}=1 \qquad \text{and} \qquad \nu_{\perp}=\tfrac{1}{2} \tag{13}$$

since

$$\langle \boldsymbol{R}(N)\rangle=N\langle \boldsymbol{r}\rangle \tag{14}$$

$$\begin{aligned}\langle \boldsymbol{R}_{\|}(N)^2\rangle&=\langle(\boldsymbol{R}(N)\cdot \boldsymbol{e})^2\rangle\\&=N(N-1)\langle \boldsymbol{r}\cdot \boldsymbol{e}\rangle^2+N\langle(\boldsymbol{r}\cdot \boldsymbol{e})^2\rangle\\&=N(N-1)\langle \boldsymbol{r}\rangle^2+N\langle(\boldsymbol{r}\cdot \boldsymbol{e})^2\rangle\end{aligned} \tag{15}$$

and

$$\langle \boldsymbol{R}_\perp(N)^2\rangle = N\{\langle r^2\rangle - \langle (\boldsymbol{r}\cdot \boldsymbol{e})^2\rangle\} \tag{16}$$

where

$$\langle r^m\rangle \equiv \int r^m p(\boldsymbol{r})\, \mathrm{d}^d r \neq 0 \qquad m = 1, 2 \tag{17}$$

are independent of N.

If $q(\boldsymbol{r}(N-k), \ldots, \boldsymbol{r}(N-1)) \neq 0$, the constraint of self-avoidance makes it more difficult to get the critical exponents of the directed self-avoiding flights. Now we first consider the case in 2D. For $r_1(N-k) = \ldots = r_1(N-1) = 0$, the steps $\boldsymbol{r}(N-k), \ldots, \boldsymbol{r}(N-1)$ belong to a 1D directed flight such that the probabilities

$$q(\boldsymbol{r}(N-k), \ldots, \boldsymbol{r}(N-1)) = \begin{cases} q_- & \text{if } r_2(N-1) > 0 \\ q_+ & \text{if } r_2(N-1) < 0 \end{cases} \tag{18}$$

depend only on the last step, where

$$q_\pm \equiv \int_{\substack{r_1=0\\ \pm r_2>0}} p(\boldsymbol{r})\, \mathrm{d}^d \boldsymbol{r} \tag{19}$$

with $q_+ + q_- + P = 1$. Let us denote the stationary probabilities for a step with $r_1 = 0$ and $\pm r_2 > 0$ by $w_\pm < q_\pm$ and that for a step with $r_1 > 0$ by $w > P$. Then these stationary probabilities can be determined as the components of the eigenvector of the transfer matrix

$$\begin{pmatrix} q_+/(1-q_-) & 0 & q_+ \\ 0 & q_-/(1-q_+) & q_- \\ P/(1-q_-) & P/(1-q_+) & P \end{pmatrix}$$

corresponding to the eigenvalue 1 with the normalization condition $w_+ + w_- + w = 1$, yielding the more difficult equation for the preferred direction

$$\lim_{N\to\infty} \frac{\boldsymbol{R}(N)}{N} = \left(w_+ \int_{\substack{r_1=0\\ r_2>0}} + w_- \int_{\substack{r_1=0\\ r_2<0}} + w \int_{r_1>0} \right) \boldsymbol{r} p(\boldsymbol{r})\, \mathrm{d}^d \boldsymbol{r} \tag{20}$$

which implies immediately $\nu = \nu_\parallel = 1$. For example, for the symmetrical case $q_+ = q_- = q$, it means

$$w_+ = w_- = \frac{q - q^2}{(1-2q^2)} \tag{21}$$

and

$$w = \frac{1-2q}{1-2q^2}. \tag{22}$$

One can show rigorously that the critical exponent $\nu_\perp$ for the Markovian flight with one-step memory remains the same as that for the independent flight though (16) has to be modified.

For $d > 2$ and $P < 1$ the directed self-avoiding flight is no longer Markovian. But the probability for k consecutive steps with $r_1(N-k) = \ldots = r_1(N-1) = 0$ decreases at least as $(1-P)^k$ so that we expect no effect on the critical exponents. Besides, the upper boundary dimension for the usual (non-directed) 'true' SAW [13] is $d_c = 2$ and the critical exponents for the directed SAW coincide with those for the directed random walk [11]. That is why we confine ourselves in the next section to the case of independent directed Lévy flights.

3. Directed Lévy flights

Let us consider the example of a probability distribution $p(\boldsymbol{r})$ with $p(\boldsymbol{r}) = 0$ for $r_1 < 0$ and

$$P(r) \equiv \int |r|^{d-1} p(\boldsymbol{r})\, \mathrm{d}\Omega = 2^{-y(\mu+1)} \qquad \text{for } 2^y \leqslant |r| < 2^{y+1} \tag{23}$$

with $0 < \mu \leqslant 2$. This probability distribution looks convenient to simulate a Lévy-like flight since the probability $P(r)$ depends only on the number of digits or r in the binary representation. But we can show that it does not belong to the domain of attraction of a Lévy distribution; nevertheless, the critical exponents exist and coincide with that of a Lévy flight with parameter μ.

Often Lévy flights are restricted to the case of a probability distribution with decay $cr^{-\mu-1}$, $\mu < 2$, for the radial probability $P(r)$ (r sufficiently large) for two different reasons:

(i) If $\mu = 2$ and $P = 1$ then the appropriately rescaled quantity $\{\boldsymbol{R}(N) - \langle \boldsymbol{R}(N)\rangle\}/B(N)$ tends (as $\{\boldsymbol{R}(N) - \langle \boldsymbol{R}(N)\rangle\}/\sqrt{N}$ for $\langle \boldsymbol{r}^2\rangle < \infty$) to a Gaussian distribution [2]. Since now $B(N) \neq \sqrt{N}$ indicating a novel critical behaviour, we will nevertheless treat the case $\mu = 2$ separately.

(ii) If $\mu < 2$ but the function $L_i(x) \equiv x^\mu \int_{|r_i| > x} p(\boldsymbol{r})\, \mathrm{d}^d\boldsymbol{r}$ does not vary slowly for large x, the probability distribution for a single component does not belong to the domain of attraction of a stable distribution [2].

Let us show that $L(x) \equiv x^\mu \int_{|\boldsymbol{r}| \geqslant x} p(\boldsymbol{r})\, \mathrm{d}^d\boldsymbol{r}$ with $p(\boldsymbol{r})$ according to (23) does not vary slowly, i.e. there is some real $c > 0$ with $\lim_{x\to\infty} L(cx)/L(x)$ non-existent or differing from 1. With $x = z \cdot 2^y$, $1 \leqslant z < 2$, we get $L(z\cdot 2^y) = z^\mu(2^{\mu+1} - 1)/(2^\mu - 1) - z^{\mu+1}$, which fluctuates in the interval

$$\left[\frac{2^\mu}{2^\mu - 1}, \frac{1}{\mu}\left(\frac{\mu(2^{\mu+1} - 1)}{(\mu+1)(2^\mu - 1)}\right)^{(1+\mu)}\right].$$

The second reason does not apply to distributions with the decay of the probability

$$\Pr\{r\colon |r| > x\} = cx^{-\mu}(\ln x)^{-\kappa} \qquad 0 < \mu < 2. \tag{24}$$

We will also consider such distributions and refine the definition of the critical exponents in order to measure the influence of κ. Further, we will also investigate the cases $\mu = 2$ with $\kappa \leqslant 1$ and $\mu = 0$ not leading to Lévy distributions.

The expectation value $\langle \boldsymbol{r}\rangle$ does not exist for $\mu < 1$ or $\mu = 1$ with $\kappa \leqslant 1$ such that a preferred direction can no longer be defined by (6). But the preferred direction for the special cases given by symmetry arguments as (5a, b) in section 2 can, of course, be used further. Then we will see below that $\boldsymbol{R}_\parallel(N)$ and $\boldsymbol{R}_\perp(N)$ do not differ in their asymptotic behaviour under mild assumptions.

Since the variance of $\boldsymbol{r}$ is diverging, the critical exponents cannot be defined with the help of (9). Then let us introduce three other definitions:

(i) Consider the mean of order α, $M_\alpha^*(N) \equiv \langle |\boldsymbol{R}_*(N)|^\alpha \rangle^{1/\alpha}$, if it exists. If for every $\varepsilon > 0$

$$\lim_{N\to\infty} M_\alpha^*(N)/\{N^{\nu_*}(\ln N)^{\lambda_*+\varepsilon}\} = 0 \tag{25}$$

and

$$\lim_{N\to\infty} M_\alpha^*(N)/\{N^{\nu_*}(\ln N)^{\lambda_*-\varepsilon}\} = \infty \tag{26}$$

we will write

$$R_*(N) \sim N^{\upsilon_*}(\ln N)^{\lambda_*} \qquad \text{for } N \to \infty \tag{27}$$

where λ_* is called the critical exponent of the logarithmic correction [7]. These means have, for non-degenerate random variables, the property that $M_\alpha^* < M_\beta^*$ if $\alpha < \beta$. In the field of ordinary random flights and of Lévy flights especially, the means with $\alpha = 2$ (quadratic mean), $\alpha = 1$ (arithmetic mean) and $\alpha = 0$ (geometric mean) [4, 7] with $M_0^*(N) = \exp\langle \ln|\boldsymbol{R}_*(N)|\rangle$ were applied; and in other papers [15] about random structures $\alpha = -1$ (harmonic mean) and $\alpha = -2$ also appeared. Immediately two questions arise:

(*a*) For which α is the mean $M_\alpha^*(N)$ finite? Since the N-fold convolution of $P^*(r)$ has the same asymptotic behaviour [2] of the tail $r^{-\mu}(\ln r)^{-\kappa}$ as $P^*(r)$, $M_\alpha^*(N)$ is finite for $\alpha < \mu$ and diverging for $\alpha > \mu$.

(*b*) If M_α^* and M_β^* are both finite do the pairs $(\upsilon_{*\alpha}, \lambda_{*\alpha})$ and $(\upsilon_{*\beta}, \lambda_{*\beta})$ coincide with each other? This question has an encouraging answer: if $P^*(r)$ belongs to the domain of attraction of a stable distribution with index μ, then $\{\boldsymbol{R}_*(N) - \boldsymbol{A}_*(N)\}/B_*(N)$ tends for a pair of appropriate sequences $\boldsymbol{A}_*(N)$ and $B_*(N)$ to this stable distribution and for $\alpha < \mu$ the mean of order α of $\{\boldsymbol{R}_*(N) - \boldsymbol{A}_*(N)\}/B_*(N)$ has a finite limit. Independent of α $(\alpha < \mu)$, the behaviour of $M_\alpha^*(N)$ is given by either $A_*(N)$ or $B_*(N)$. Hence the use of the geometric mean [4, 7] contains no arbitrariness.

If $\mu > 1$ or $\mu = 1$ with $\kappa > 1$, then $\boldsymbol{A}_*(N) = \langle \boldsymbol{R}_*(N)\rangle = N\langle \boldsymbol{r}_*\rangle$ determines the behaviour of the mean $M_\alpha^*(N)$ as

$$M_\alpha^*(N) \sim N \qquad \text{for } \langle r_*\rangle \neq 0 \tag{28}$$

whereas $B_*(N)$ does as

$$M_\alpha^*(N) \sim B_*(N) \qquad \text{for } \langle r_*\rangle = 0 \tag{29}$$

with [2]

$$B_*(N) \sim \begin{cases} N^{1/\mu}(\ln N)^{-\kappa/\mu} & \text{for } \mu < 2 \\ N^{1/2}(\ln N)^{(1-\kappa)/2} & \text{for } \mu = 2, \kappa < 1 \\ N^{1/2}(\ln(\ln N))^{1/2} & \text{for } \mu = 2, \kappa = 1. \end{cases} \tag{30}$$

If $\mu = 1$ with $\kappa \leq 1$, then $B_*(N) \sim N(\ln N)^{-\kappa}$ does not influence the critical behaviour of $M_\alpha^*(N)$, hence

$$M_\alpha^*(N) \sim A_*(N) \sim \begin{cases} N(\ln N)^{1-\kappa} & \text{for } \kappa < 1 \\ N\ln(\ln N) & \text{for } \kappa = 1. \end{cases} \tag{31}$$

If $\mu < 1$, then $\boldsymbol{A}_*(N) = 0$ and

$$M_\alpha^*(N) \sim B_*(N) \sim N^{1/\mu}(\ln N)^{-\kappa/\mu}. \tag{32}$$

For $\kappa = 0$, the results by Lee *et al* [7] for the case $\mu = 2$ are confirmed.

If $P^*(\boldsymbol{r})$ does not belong to the domain of attraction of a stable distribution as our example given by (23), then include it for the case $\langle \boldsymbol{r}_* \rangle = 0$ or non-existent but varying between two distributions $P_1^*(\boldsymbol{r})$ and $P_2^*(\boldsymbol{r})$ with $P_1^*(\boldsymbol{r}) \leqslant P^*(\boldsymbol{r}) \leqslant P_2^*(\boldsymbol{r})$ for sufficiently large $|\boldsymbol{r}|$, where P_1^* and P_2^* lie in the domain of attraction of Lévy distributions with the same normalizing constant $B_*(N)$. We will assume that this is always possible, excluding probability distributions with a too strong variation of $L(x)$. The above idea is only a plausibility argument for $M_\alpha^*(N) \sim B_*(N)$ with $*$ representing $\perp$ or no index. For $* = \|$ it can be extended to a rigorous proof.

(ii) The answer to question (i*a*) can be exploited for the following crude definition which does not specify λ:

$$v_\| = \min\{1, \min\{\alpha^{-1}: M_\alpha^\|(N) < \infty\}\} \tag{33}$$

and for $1 < \mu \leqslant 2$

$$v_\perp = \min\{\alpha^{-1}: M_\alpha^\perp(N) < \infty\}. \tag{34}$$

This definition is an analogy to the definition of the Hausdorff dimension of a fractal [16].

A fine classification would be possible with the help of refined means. Since we are not aware of such means in the literature, let us introduce the means

$$M_{\alpha,\beta}(N) = f_{\alpha,\beta}^{-1}\{\langle f_{\alpha,\beta}(|\boldsymbol{R}(N)|)\rangle\} \tag{35}$$

with

$$f_{\alpha,\beta}(x) = |x|^\alpha\{\ln(1+|x|)\}^\beta \qquad \alpha > 0. \tag{36}$$

The analogy to the refined Hausdorff dimension according to Hausdorff's paper from 1918 [16] is obvious, only the asymptotic regions differ: Hausdorff has to consider the behaviour of $f_{\alpha,\beta}$ for $x \to 0$, whereas for our purpose the behaviour for $x \to \infty$ has to be considered. The refined Hausdorff dimension seldom appears in the modern literature about fractals though it is useful for very simple examples (cf appendix).

For large $|x|$ the iterated function $f_{\gamma,\delta}\{f_{\alpha,\beta}(x)\}$ is approximately $f_{\alpha\gamma,\beta\gamma+\delta}(x)$, i.e.

$$f_{\alpha,\beta}^{-1}(x) \sim f_{1/\alpha,\beta/\alpha}(x). \tag{37}$$

The second derivative of $f_{\alpha,\beta}(x)$ is approximately

$$x^{\alpha-2}(\ln(1+x))^{\beta-2}\{\alpha(\alpha-1)\ln^2(1+x) + 2\alpha\beta x^{\alpha-2}(\ln(1+x))^{\beta-1}\}$$

for large x, i.e. $f_{\alpha,\beta}(x)$ is concave for $0 < \alpha < 1$ or $\alpha = 1$ with $\beta < 0$ and convex otherwise. Jensen's inequality implies that, for large $R(N)$, $M_{\varepsilon,\eta}^*(N) < M_{1,0}^*(N)$ for $\varepsilon < 1$ or $\varepsilon = 1$ with $\eta < 0$. The approximate iteration rule finally yields $M_{\alpha,\beta}^*(N) < M_{\gamma,\delta}^*(N)$ for large $R(N)$, $\alpha < \gamma$ or $\alpha = \gamma$ with $\beta < \delta$ (by taking $\varepsilon = \alpha/\gamma$ and $\eta = \beta - \alpha\delta/\gamma$). Since the N-fold convolution of $P^*(r)$ has the same asymptotic behaviour of the tail $r^{-\mu-1}(\ln r)^{-\kappa}$, $M_{\alpha,\beta}^*$ is obviously finite if and only if $\alpha < \mu$ or $\alpha = \mu$ with $\beta < \kappa - 1$. The comparison with definition (i) yields

$$\lambda_\| = 0 \qquad \text{for } \min\{\alpha^{-1}: M_{\alpha,\lambda}^\|(N) < \infty\} > 1 \tag{38}$$

and

$$\lambda_\| = v_\|(1 + \max\{\beta: M_{1/v_\|,\beta}^\| < \infty\}) \tag{39}$$

for $\min\{\alpha^{-1}: M_{\alpha,\lambda}^\|(N) < \infty\} < 1$.

(iii) The following definition is mostly appropriate for computer simulation. Now for fixed number of steps N typical Lévy flights are filtered out, i.e. those with $|r_i| < F(N)$

for every i $(1<i<d)$. The largest step-length $F(N)$ is chosen in such a way that the probability

$$\Pr\{r(k), \forall k \leqslant N: |r(k)| < F(N)\} \sim 1 - \exp\{-N \cdot \{F(N)\}^{-\mu}\{\ln F(N)\}^{-\kappa}\}$$

is of order O(1), i.e.

$$F(N) \sim f_{1/\mu, -\kappa/\mu}(N) \qquad \text{for } \mu > 0 \tag{40}$$

and

$$F(N) \sim \exp\{N^{1/\kappa}\} \qquad \text{for } \mu = 0. \tag{41}$$

Then the averages $\langle \boldsymbol{R}_*(N)\rangle_{F(N)}$ and $\langle \boldsymbol{R}_*^2(N)\rangle_{F(N)}$ can be easily calculated, where the index $F(N)$ means that only typical flights are averaged. As an example, we consider the simulation by Hu and Yao [12]: $N \simeq 100$ and $0.5 \leqslant \mu \leqslant 1.6$ mean that $F(100)$ is of order between 10 000 and 18. The cut-off $F(N) = 6$ in [12] is obviously much too small, hence the result of their simulation is not surprising. They get the critical exponents of an ordinary flight with finite support of $p^*(r)$.

Now let us calculate the critical exponents for this definition: if $\langle \boldsymbol{r}\rangle$ exists, i.e. for $\mu > 1$ or $\mu = 1$ with $\kappa \leqslant 1$, we have for large N

$$\langle \boldsymbol{R}_\perp^2(N)\rangle_{F(N)}^{1/2} \sim N^{1/2}(\langle \boldsymbol{r}_\perp^2\rangle_{F(N)})^{1/2}$$
$$\sim \begin{cases} N^{1/\mu}(\ln N)^{-\kappa/\mu} & \text{for } 1<\mu<2 \\ N^{1/2}(\ln(\ln N))^{1/2} & \text{for } \mu = 2 \text{ with } \kappa = 1 \\ N^{1/2}(\ln N)^{(1-\kappa)/2} & \text{for } \mu = 2 \text{ with } \kappa < 1 \end{cases} \tag{42}$$

while

$$\langle R(N)\rangle \sim N \tag{43}$$

in coincidence with definition (i), and otherwise

$$\langle \boldsymbol{R}(N)\rangle_{F(N)} \sim N\langle \boldsymbol{r}\rangle_{F(N)}$$
$$\sim \begin{cases} N^{1/\mu}(\ln N)^{-\kappa/\mu} & \text{for } 0<\mu<1 \\ N(\ln N)^{1-\kappa} & \text{for } \mu = 1 \text{ with } \kappa < 1 \\ N \ln(\ln N) & \text{for } \mu = 1 \text{ with } \kappa = 1 \\ N^{-1/\kappa}\exp(N^{1/\kappa}) & \text{for } \mu = 0. \end{cases} \tag{44}$$

Again the critical exponents coincide with those from definition (i). A slowly varying function $L(x)$ does not pose any additional difficulties. This cut-off method can be applied to the 2D DSALF, where the modified equation (20) shows that the critical exponents are not changed due to the self-avoidance condition.

It is an open problem to show that the critical behaviour according to definitions (i) and (iii) coincides if an arbitrary finer classification like $N^\nu(\ln N)^\lambda(\ln(\ln N))^\omega$ is introduced.

Appendix

Consider Pascal's triangle modulo 4, where every binomial coefficient divisible by 4 is represented by a white pixel and the others by a black one. Let V be the limit $n \to \infty$ of 2^n lines of that triangle reduced to a triangle of fixed size. V consists of three copies of itself of half size and a Sierpinski gasket S around the centre of the triangle. S consists of three copies of itself of half size. If the three copies and S are decomposed again then one gets nine copies of V of a quarter size and six copies of S of half size. By induction one can show that after n such steps V is decomposed into 3^n copies of V with similarity factor 2^{-n} and $n \cdot 3^{n-1}$ copies of S with factor 2^{1-n}. Hence the Hausdorff dimension is $\ln 3/\ln 2$ which can also be concluded from theorem 4 in [17]. But the $n \cdot 3^{n-1}$ copies of S imply that this dimension has a logarithmic refinement in the first power, which has perhaps not been noted before.

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